

The pionium lifetime in generalized chiral perturbation theory

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Abstract

Pionium lifetime corrections to the nonrelativistic formula are calculated in the framework of the quasipotential–constraint theory approach. The calculation extends an earlier evaluation, made in the scheme of standard chiral perturbation theory, to the scheme of generalized chiral perturbation theory, in which the quark condensate is left as a free parameter. The pionium lifetime is calculated as a function of the combination $(a_0^0 - a_0^2)$ of the $\pi\pi$ S -wave scattering lengths with isospin $I = 0, 2$.

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The DIRAC experiment at CERN [1] is expected to allow, in the near future, the measurement of the lifetime of the pionium ($\pi^+\pi^-$ atom) with a 10% precision. The latter, in turn, through its relationship with $\pi\pi$ scattering lengths, would provide a determination of the combination $(a_0^0 - a_0^2)$ of the S -wave scattering lengths with isospin $I = 0, 2$ with 5% accuracy. The strong interaction scattering lengths a_0^0 and a_0^2 have been evaluated in the literature in the framework of chiral perturbation theory (χPT) to two-loop order of the chiral effective lagrangian [2, 3, 4]. Therefore, the pionium lifetime measurement provides a high precision experimental test of chiral perturbation theory predictions.

The nonrelativistic formula of the pionium lifetime in lowest order of electromagnetic interactions was first evaluated by Deser *et al.* [5] and later reanalyzed by others [6]. It reads:

$$\frac{1}{\tau_0} = \Gamma_0 = \frac{16\pi}{9} \sqrt{\frac{2\Delta m_\pi}{m_{\pi^+}}} \frac{(a_0^0 - a_0^2)^2}{m_{\pi^+}^2} |\psi_{+-}(0)|^2, \quad \Delta m_\pi = m_{\pi^+} - m_{\pi^0}, \quad (1)$$

where $\psi_{+-}(0)$ is the wave function of the pionium at the origin (in x -space).

An evaluation of the relativistic and higher-order electromagnetic corrections to the above formula was recently done by several authors. In the frameworks of quantum field theory and χPT , three different methods of evaluation have led to similar estimates, of the order of 6%, of these corrections [7, 8, 9]. The first method uses a three-dimensionally reduced form of the Bethe–Salpeter equation (the quasipotential–constraint theory approach) and deals with an off-mass shell formalism [7]. The second method uses the Bethe–Salpeter equation with the Coulomb gauge [8]. The third one uses the approach of nonrelativistic effective theory [9, 10, 11, 12, 13]. (A survey of other approaches is presented in the second paper of Ref. [9].)

The pionium lifetime, with the sizable relativistic and electromagnetic corrections included in, can be represented as:

$$\begin{aligned} \frac{1}{\tau} = \Gamma &= \frac{1}{64\pi m_{\pi^+}^2} (\text{Re} \widetilde{\mathcal{M}}_{00,+})^2 (1 + \gamma) |\psi_{+-}(0)|^2 \sqrt{\frac{2\Delta m_\pi}{m_{\pi^+}} \left(1 - \frac{\Delta m_\pi}{2m_{\pi^+}}\right)} \\ &\equiv \Gamma_0 \sqrt{\left(1 - \frac{\Delta m_\pi}{2m_{\pi^+}}\right)} \left(1 + \frac{\Delta\Gamma}{\Gamma_0}\right), \end{aligned} \quad (2)$$

where $\text{Re} \widetilde{\mathcal{M}}_{00,+}$ is the real part of the on-mass shell scattering amplitude of the process $\pi^+\pi^- \rightarrow \pi^0\pi^0$, calculated at threshold, in the presence of electromagnetic interactions and from which singularities of the infra-red photons have been appropriately subtracted [14]; the factor γ represents the contribution of interactions at second-order of perturbation theory with respect to the nonrelativistic zeroth-order Coulomb hamiltonian. The explicit expressions of $\text{Re} \widetilde{\mathcal{M}}_{00,+}$ and of γ may differ from one approach to the other, according to the way the singularities of the infra-red photons are subtracted, but their total contribution should be the same.

The purpose of the present article is to calculate the pionium lifetime in the framework of generalized χPT [15, 4]. In this framework, according to the observation that the fundamental order parameter of spontaneous chiral symmetry breaking in QCD is F_π , the decay coupling constant of the pion, the quark condensate in the chiral limit $\langle 0|\bar{q}q|0 \rangle_0$ is left as a free parameter. Its value should depend on the details of the mechanism of chiral symmetry breaking and would require an independent experimental test. In the framework of standard χPT , the mechanism of chiral symmetry breaking would be very similar to that realized in a ferromagnetic medium [16]. In this case, the value of the Gell-Mann–Oakes–Renner (GOR) parameter [17], defined as

$$x_{GOR} = -\frac{2\hat{m} \langle 0|\bar{q}q|0 \rangle_0}{F_\pi^2 m_\pi^2}, \quad (3)$$

where $2\hat{m} = m_u + m_d$ and m_π is the pion physical mass, would be close to one. Equiva-

lently, the quark condensate parameter

$$B \equiv -\frac{\langle 0|\bar{q}q|0\rangle_0}{F^2}, \quad (4)$$

where F is F_π in the chiral $SU(2) \times SU(2)$ limit, would be of the order of the hadronic mass scale $\Lambda_H \sim 1$ GeV. This assumption fixes the way standard χPT is expanded: the quark condensate parameter B is assigned dimension zero in the infra-red external momenta of the Goldstone bosons, while the quark masses are assigned dimension two [2].

On the other hand, in an antiferromagnetic medium, one meets a situation where the quark condensate would be zero or very small [16, 18]. Recently other possibilities were also advocated [19, 20]: the existence of a possible chiral phase transition in QCD [21, 22] at relatively low values of the light quark flavor number might induce, in the standard case, a strong flavor dependence of the quark condensate, which would have the tendency to decrease by passing from $SU(2) \times SU(2)$ to $SU(3) \times SU(3)$.

The framework of generalized χPT offers the possibility of experimentally testing various issues of chiral symmetry breaking mechanism. In this framework one relaxes the GOR assumption and treats the order of magnitude of the quark condensate parameter B as an *a priori* unknown quantity (awaiting a precise experimental information about it) leaving to it the possibility of reaching small or vanishing values. To this aim, B is assigned dimension one in the infra-red momenta of the external Goldstone bosons and accordingly quark masses are also assigned dimension one. Due to this rule, at each order of the perturbative expansion, generalized χPT contains more terms than standard χPT . For instance, the pion mass formula becomes at leading order:

$$(m_\pi^2)_0 = 2\hat{m}B + 4\hat{m}^2A, \quad (5)$$

where the constant A is expressible in terms of two-point functions of scalar and pseudoscalar quark densities. In the standard χPT case, this term is relegated to the next-to-leading order.

Processes involving only Goldstone bosons are sensitive at leading order to the value of the quark condensate parameter B . Among them, the $\pi\pi$ scattering amplitude plays a key role. At the tree level of the chiral effective lagrangian, the amplitude $A(s|t, u)$ has the expression:

$$A(s|t, u) = \frac{1}{F^2}(s - 2\hat{m}B), \quad (6)$$

which displays explicit dependence on the quark condensate parameter.

At the one-loop order the strong interaction on-mass shell scattering amplitude can be described by four parameters, α , β , λ_1 and λ_2 , and takes the following form [4, 23]:

$$A(s|t, u) = \mathcal{M}_{00,+-}^{str.} = \frac{\beta}{F_\pi^2}(s - \frac{4}{3}m_\pi^2) + \alpha \frac{m_\pi^2}{3F_\pi^2}$$

$$\begin{aligned}
& + \frac{\lambda_1}{F_\pi^4}(s - 2m_\pi^2)^2 + \frac{\lambda_2}{F_\pi^4}[(t - 2m_\pi^2)^2 + (u - 2m_\pi^2)^2] \\
& + \frac{1}{2F_\pi^4}[s^2 - m_\pi^4 + 8(4\hat{m}^2 A)s - 6(4\hat{m}^2 A)m_\pi^2 + 7(4\hat{m}^2 A)^2]\bar{J}(s) \\
& + \frac{1}{F_\pi^4}[(m_\pi^2 + 4\hat{m}^2 A - \frac{1}{2}t)^2 + \frac{1}{12}(s - u)(t - 4m_\pi^2)]\bar{J}(t) \\
& + \frac{1}{F_\pi^4}[(m_\pi^2 + 4\hat{m}^2 A - \frac{1}{2}u)^2 + \frac{1}{12}(s - t)(u - 4m_\pi^2)]\bar{J}(u), \tag{7}
\end{aligned}$$

where m_π is the physical pion mass, \bar{J} the conventional loop function [2] and $4\hat{m}^2 A$ the quadratic mass term present at the tree level in m_π^2 [Eq. (5)]. The parameters λ_1 and λ_2 are related to the standard renormalized low energy constants l_1^r and l_2^r [2]; their expressions (at the one-loop order) are:

$$\lambda_1 = 2l_1^r - \frac{1}{48\pi^2} \ln\left(\frac{m_\pi^2}{\mu^2}\right) - \frac{1}{36\pi^2}, \tag{8}$$

$$\lambda_2 = l_2^r - \frac{1}{48\pi^2} \ln\left(\frac{m_\pi^2}{\mu^2}\right) - \frac{5}{288\pi^2}. \tag{9}$$

(μ is the renormalization mass.) β is essentially related to the deviation of F_π from F . α is mainly related to the quark condensate parameter; its expression at the tree level is:

$$(\alpha)_0 = 1 + 3 \times \frac{4\hat{m}^2 A}{(m_\pi^2)_0}. \tag{10}$$

The one-loop expressions of F_π^2 , m_π^2 , β and α are given by the following relations:

$$F_\pi^2 = F^2 \left[1 + 2l_4^r \frac{2\hat{m}B}{F^2} + j_1^r \frac{4\hat{m}^2 A}{F^2} - \frac{2m_\pi^2}{16\pi^2} \ln\left(\frac{m_\pi^2}{\mu^2}\right) \right], \tag{11}$$

$$\begin{aligned}
m_\pi^2 &= 2\hat{m}B + 4\hat{m}^2 A + 2l_3^r \frac{(2\hat{m}B)^2}{F^2} \\
&+ j_2^r (2\hat{m}B) \frac{4\hat{m}^2 A}{F^2} + j_3^r \frac{(4\hat{m}^2 A)^2}{F^2} \\
&- \frac{2}{16\pi^2 F^2} (2\hat{m}B + 4\hat{m}^2 A) \left(-\frac{1}{4}(2\hat{m}B) + 4\hat{m}^2 A \right) \ln\left(\frac{m_\pi^2}{\mu^2}\right), \tag{12}
\end{aligned}$$

$$\frac{\beta}{F_\pi^2} = \frac{1}{F^2} \left[1 - \frac{2m_\pi^2}{16\pi^2 F^2} - \frac{4\hat{m}^2 A}{F^2} \left(j_4^r + \frac{5}{16\pi^2} (1 + \ln\left(\frac{m_\pi^2}{\mu^2}\right)) \right) \right], \tag{13}$$

$$\begin{aligned}
\alpha \frac{m_\pi^2}{F_\pi^2} &= \frac{1}{F^2} (2\hat{m}B + 4(4\hat{m}^2 A)) + 8l_3^r \frac{(2\hat{m}B)^2}{F^4} \\
&+ j_5^r \frac{1}{F^4} (2\hat{m}B)(4\hat{m}^2 A) + j_6^r \frac{(4\hat{m}^2 A)^2}{F^4} \\
&- \frac{1}{16\pi^2 F^4} \left[-2m_\pi^4 + \frac{27}{2}m_\pi^2(4\hat{m}^2 A) + \frac{33}{2}(4\hat{m}^2 A)^2 \right] \ln\left(\frac{m_\pi^2}{\mu^2}\right) \\
&- \frac{1}{16\pi^2 F^4} \left[\frac{1}{2}m_\pi^4 + 11m_\pi^2(4\hat{m}^2 A) + \frac{33}{2}(4\hat{m}^2 A)^2 \right], \tag{14}
\end{aligned}$$

where j_i^r are combinations of renormalized low energy constants present in the generalized χPT lagrangian [4, 23, 24] and l_3^r and l_4^r are the renormalized low energy constants already present in the standard χPT lagrangian [2]. [Our effective lagrangian slightly differs from that of Refs. [23, 24] in that we continue using in it the l_4 -term, instead of replacing it through the equations of motion with the $\xi^{(2)}$ -term.]

In terms of the above parameters the expressions of the S - and P -wave scattering lengths are:

$$a_0^0 = \frac{1}{96\pi} \frac{m_\pi^2}{F_\pi^2} (5\alpha + 16\beta) + \frac{5}{8\pi} \frac{m_\pi^4}{F_\pi^4} (\lambda_1 + 2\lambda_2) + \frac{1}{4608\pi^3} \frac{m_\pi^4}{F_\pi^4} (5\alpha + 16\beta)^2, \quad (15)$$

$$a_0^2 = \frac{1}{48\pi} \frac{m_\pi^2}{F_\pi^2} (\alpha - 4\beta) + \frac{1}{4\pi} \frac{m_\pi^4}{F_\pi^4} (\lambda_1 + 2\lambda_2) + \frac{1}{1152\pi^3} \frac{m_\pi^4}{F_\pi^4} (\alpha - 4\beta)^2, \quad (16)$$

$$a_1^1 = \frac{1}{24\pi} \frac{1}{F_\pi^2} \beta - \frac{1}{6\pi} \frac{m_\pi^2}{F_\pi^4} (\lambda_1 - \lambda_2) + \frac{1}{41472\pi^3} \frac{m_\pi^2}{F_\pi^4} (5\alpha^2 - 40\alpha\beta - 16\beta^2). \quad (17)$$

At the two-loop order, the $\pi\pi$ scattering amplitude is described by six parameters, α , β , λ_1 , λ_2 , λ_3 and λ_4 . The five parameters other than α are weakly dependent on the quark condensate; thus α remains the only parameter to be strongly sensitive to the quark condensate value. It is evident from the tree level relation (10) that in the standard case α remains close to 1, while in the extreme case of generalized χPT (antiferromagnetic case), where the quark condensate vanishes ($2\hat{m}B \simeq 0$), it approaches the value 4. The expressions of the scattering lengths and of the effective ranges at the two-loop order in terms of the above six parameters can be found in Ref. [4].

In order to obtain the pionium lifetime, one first needs to calculate $\mathcal{Re}\widetilde{\mathcal{M}}_{00,+}$ [Eq. (2)]. This includes, in addition to the strong interaction contributions, those of the electromagnetic interactions. These effects are taken into account in the chiral effective lagrangian by the presence of new terms involving low energy constants [25, 26, 14]. In generalized χPT the effective lagrangian contains more terms than the standard one, these being proportional to the low energy parameter A [Eqs. (5) and (10)]. The additional electromagnetic terms needed for the present process are given, in standard notations, by the following part of the effective lagrangian (in the $SU(2) \times SU(2)$ case):

$$\begin{aligned} \mathcal{L}^{(G\chi PT)} = \frac{1}{2} F^2 A \Big\{ & k_{15} \langle QUQU^\dagger \rangle \langle \chi^\dagger U + U^\dagger \chi \rangle^2 \\ & + 8k_{16} \langle QUQU^\dagger \rangle \langle \chi^\dagger \chi \rangle + k_{17} \langle Q^2 \rangle \langle \chi^\dagger U + U^\dagger \chi \rangle^2 \\ & + k_{19} \langle (QU\chi^\dagger U + U^\dagger \chi U^\dagger Q)^2 \rangle + k_{20} \langle QU\chi^\dagger U \rangle \langle U^\dagger \chi U^\dagger Q \rangle \Big\}, \quad (18) \end{aligned}$$

where Q is the quark charge matrix, $Q = e \times \text{diag.}(2/3, -1/3)$, and the sources χ have been restricted to a combination of an isosinglet scalar density and an isotriplet of pseudoscalar densities, $\chi = s + i\mathbf{p} \cdot \boldsymbol{\tau}$, with s and \mathbf{p} real, $\boldsymbol{\tau}$ representing the Pauli matrices. The renormalization of the coefficients k is done, with dimensional regularization in d -dimensional

space-time, according to the decomposition

$$k_i = \kappa_i \lambda + k_i^r(\mu), \quad \lambda = \frac{\mu^{d-4}}{16\pi^2} \left(\frac{1}{d-4} - \frac{1}{2} (\ln(4\pi) + \Gamma'(1) + 1) \right), \quad (19)$$

μ being the renormalization mass. The μ -dependence of the renormalized coefficients $k_i^r(\mu)$ is fixed by the prescription that the k s are μ -independent. The coefficients κ are found to have the following values: $\kappa_{15} = \frac{1}{2} + 6Z$, $\kappa_{16} = -Z$, $\kappa_{17} = -\frac{1}{2} - \frac{12}{5}Z$, $\kappa_{19} = 2$, $\kappa_{20} = 5$, where $Z = C/F^4$, C being the coefficient of the lowest-order, $O(e^2 p^0)$, electromagnetic term

$$\mathcal{L}^{(e^2 p^0)} = C \langle QUQU^\dagger \rangle, \quad (20)$$

responsible for the pion mass shift at that order [27]:

$$(\Delta m_\pi^2)_0 = 2e^2 \frac{C}{F^2}, \quad \Delta m_\pi^2 \equiv m_{\pi^+}^2 - m_{\pi^0}^2. \quad (21)$$

We use in the following for the other coefficients k the notations of Ref. [26].

Treating the electromagnetic interaction in $\mathcal{R}\widetilde{\mathcal{M}}_{00,+ -}$ as a perturbation yields for the zeroth-order part of the latter the strong interaction amplitude, which is already calculated in generalized χPT to two-loop order [4]. It is therefore sufficient to calculate, up to one-loop order of the chiral effective lagrangian, the first-order electromagnetic correction to it. It should be emphasized here that, at the numerical level, the strong interaction amplitude is calculated in the literature [2, 3, 4] by identifying the pion mass appearing in it with the physical mass of the charged pion.

The electromagnetic corrections can be divided into two categories. The first one concerns corrections that arise from conventional pion-photon interactions. The second one concerns corrections that arise from quark-photon interactions which manifest themselves in the chiral effective lagrangian through the presence of the $O(e^2 p^0)$ mass shift term of the charged pion [Eq. (20)], which survives the chiral limit and produces the main part of the pion mass difference [27]. (It also induces, through renormalization, higher-order counterterms in the chiral effective lagrangian.)

The pion-photon interaction yields in general an infra-red divergent amplitude on the mass-shell. Usually, this divergence is avoided by giving the photon a small mass and then subtracting the singular pieces [14]. In our approach, based on the quasipotential-constraint theory method [7], we use an off-mass shell formalism, where the total energy of the two-pion system is fixed at the bound state energy, that is, below threshold. In this case the photon field can be taken massless and the corresponding scattering amplitude is finite. However, it still contains a spurious finite infra-red term (with respect to the bound state problem). This term is cancelled by the presence of three-dimensional diagrams, called constraint diagrams, the role of which is to transform, through the Lippmann-Schwinger equation, the scattering amplitude into an irreducible kernel or a potential.

The sum of the two contributions is finite and free of spurious terms [7]. It then can be continued, neglecting $O(e^4)$ terms, to the threshold of the $\pi^+\pi^-$ system.

The quark-photon interaction terms are free of infra-red singularities and the corresponding part of the scattering amplitude can be readily calculated, taking into account in particular the pion mass difference.

The electromagnetic corrections to the neutral and charged pion masses are given to order e^2p^2 by the following formulas:

$$\begin{aligned}
m_{\pi^0}^2 &= m_\pi^2 + 2e^2 \frac{C}{16\pi^2 F^4} (2\hat{m}B)(1 + \ln(\frac{m_\pi^2}{\mu^2})) \\
&\quad + e^2 (2\hat{m}B) \left(-\frac{20}{9}(k_2^r + k_{10}^r) + 2(2k_3^r + k_4^r) + \frac{20}{9}(k_7^r + k_{11}^r) \right) \\
&\quad + e^2 (4\hat{m}^2 A) \left(-\frac{20}{9}(k_2^r + k_{10}^r) + 2(2k_3^r + k_4^r) + \frac{20}{9}(k_{15}^r + k_{17}^r + k_{19}^r) - \frac{8}{9}k_{20}^r \right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
m_{\pi^+}^2 &= m_{\pi^0}^2 + 2e^2 \frac{C}{F^2} + e^2 \frac{m_\pi^2}{16\pi^2} (7 - 3\ln(\frac{m_\pi^2}{\mu^2})) \\
&\quad - 2e^2 \frac{C}{16\pi^2 F^4} \left[(2\hat{m}B)(1 + 3\ln(\frac{m_\pi^2}{\mu^2})) + (4\hat{m}^2 A)(4 + 6\ln(\frac{m_\pi^2}{\mu^2})) \right] \\
&\quad + e^2 (2\hat{m}B) \left(-2(2k_3^r + k_4^r) + 2(k_7^r - 2k_8^r) + 6(k_7^r + 2k_8^r) \right) \\
&\quad + e^2 (4\hat{m}^2 A) \left(-2(2k_3^r + k_4^r) + 4k_{15}^r + 4k_{16}^r - 2k_{19}^r + k_{20}^r \right),
\end{aligned} \tag{23}$$

where m_π is the strong interaction mass (12).

We present in the following the difference, $\Delta\mathcal{Re}\mathcal{M}_{00,+ -}$, at the $\pi^+\pi^-$ threshold, of $\mathcal{Re}\widetilde{\mathcal{M}}_{00,+ -}$ [Eq. (2)] calculated in the presence of electromagnetism including the terms of order e^2p^0 and e^2p^2 and of the strong interaction amplitude $\mathcal{Re}\mathcal{M}_{00,+ -}^{str.}$ [Eq. (7)] calculated with the physical charged pion mass:

$$\begin{aligned}
\Delta\mathcal{Re}\mathcal{M}_{00,+ -} &= 4\left(\frac{F_\pi^2}{F^2} - \beta\right)\frac{\Delta m_\pi^2}{F_\pi^2} + \left(2(4\beta - \alpha) - (4 - (\alpha)_0)\right)\frac{\Delta m_\pi^2}{3F_\pi^2} \\
&\quad + \frac{m_\pi^2 \Delta m_\pi^2}{16\pi^2 F_\pi^4} \left[\left(\frac{4\beta - \alpha}{3}\right)^2 (1 + \ln(\frac{m_\pi^2}{\mu^2})) - \frac{2\beta}{3}(8\beta + \alpha) \ln(\frac{m_\pi^2}{\mu^2}) \right] \\
&\quad - e^2 \frac{m_\pi^2}{48\pi^2 F_\pi^2} (8\beta + \alpha)(5 + 3\ln(\frac{m_\pi^2}{\mu^2})) \\
&\quad + \frac{m_\pi^2 \Delta m_\pi^2}{F_\pi^4} \beta \left[4\beta\lambda_1 - \frac{2}{3}(\alpha - \beta) \left(\frac{1}{2}a_2^r + b_1^r\right) \right] \\
&\quad + \frac{m_\pi^2 \Delta m_\pi^2}{16\pi^2 F_\pi^4} [9\beta^2 - 8\alpha\beta - 2\alpha^2 - \frac{1}{27}(7\beta^2 + 10\alpha\beta + 7\alpha^2)] \\
&\quad + 2\beta e^2 \frac{m_\pi^2}{F_\pi^2} \left\{ \left[\frac{1}{3}(2k_2^r - 10k_{10}^r) - 2(2k_3^r + k_4^r) + (k_7^r - 2k_8^r) + 3(k_7^r + 2k_8^r)\right] \right. \\
&\quad \left. + \frac{(\alpha - \beta)}{3\beta} \left[\frac{10}{9}(k_2^r + k_{10}^r) - \frac{10}{9}(k_7^r + k_{11}^r) - (k_7^r - 2k_8^r)\right] \right\}
\end{aligned}$$

$$\begin{aligned}
& -3(k_7^r + 2k_8^r) - \frac{1}{9}(46k_{15}^r + 10k_{17}^r + 32k_{19}^r - 11k_{20}^r)] \\
& + \frac{2}{9} \left(\frac{\alpha - \beta}{3\beta} \right)^2 [5(k_7^r + k_{11}^r) - 5(k_{15}^r + k_{17}^r + k_{19}^r) + 2k_{20}^r] \}. \quad (24)
\end{aligned}$$

Here $(\alpha)_0$ represents α in the chiral $SU(2) \times SU(2)$ limit [Eq. (10)]; m_π and F_π correspond to the strong interaction quantities [Eqs. (11)-(12)]; a_2^r and b_1^r are low energy constants appearing in the generalized χPT lagrangian [23, 24]; $\Delta m_\pi^2 = m_{\pi^+}^2 - m_{\pi^0}^2$.

We list below the remaining corrections to the pionium decay width [7]. (We designate by α^{em} the fine structure constant in order to distinguish it from the parameter α introduced previously.)

1) The strong interaction correction coming from second order perturbation theory with respect to the bound state wave equation is:

$$(\Delta\Gamma)_{str.} = 1.5\alpha^{em}(2a_0^0 + a_0^2)\Gamma_0. \quad (25)$$

2) The vacuum polarization correction is:

$$(\Delta\Gamma)_{vac. pol.} = 0.41\alpha^{em}\Gamma_0. \quad (26)$$

3) An effective $O(e^2p^2)$ correction also arises from formal $O(e^2p^4)$ effects, corresponding to diagrams with one pion loop and one photon propagator. In the present bound state formalism, the diagram with one pion loop with one photon exchange provides an infra-red logarithmic contribution, which is:

$$\begin{aligned}
(\Delta\Gamma)_{O(e^2p^4)} &= -\frac{\alpha^{em}}{3}(2a_0^0 + a_0^2)(2\ln\alpha^{em} + 3\ln 2 + 21\zeta(3)/(2\pi^2))\Gamma_0 \\
&= 2.2\alpha^{em}(2a_0^0 + a_0^2)\Gamma_0. \quad (27)
\end{aligned}$$

This effect could be considered as being part of $\mathcal{R}e\widetilde{\mathcal{M}}_{00,+}$ [Eq. (2)]; however, since $\Delta\mathcal{R}e\mathcal{M}_{00,+}$ in Eq. (24) has been defined as including only one-loop graphs, we write it here separately.

4) The corrections due to isospin breaking in the quark masses being quadratic in $(m_u - m_d)$ [2], provide negligible effects and are ignored.

5) The kinematic correction coming from the phase space factor [Eq. (2)] is not included in the definition of $\Delta\Gamma/\Gamma_0$ and should be directly incorporated in Γ or τ .

Collecting all these contributions, one finds for the total dynamical correction to the nonrelativistic decay width formula [Eqs. (1)-(2)] the following expression:

$$\frac{\Delta\Gamma}{\Gamma_0} = \frac{6\Delta\mathcal{R}e\mathcal{M}_{00,+}}{32\pi(a_0^0 - a_0^2)} + 3.7\alpha^{em}(2a_0^0 + a_0^2) + 0.41\alpha^{em}, \quad (28)$$

where a_0^0 and a_0^2 are the strong interaction scattering lengths calculated up to two loops in the framework of generalized χPT [4].

In order to evaluate the pionium lifetime, one first needs to know the value of the combination $(a_0^0 - a_0^2)$ of the scattering lengths, which enters in the nonrelativistic formula (1) as well as in the correction (28). The evaluation of the latter necessitates also the knowledge of the other parameters of the theory. Among the six parameters $(\alpha, \beta, \lambda_i, i = 1, \dots, 4)$ of the strong interaction amplitude five of them (β and the λ s) are weakly dependent on the quark condensate value. It is then natural to fix them at some mean values and to consider α as the only variable in the preceding relations. The precise values of the six parameters depend on the behavior of the $\pi - \pi$ scattering amplitude in the whole low energy kinematic region. A determination of these values in the standard χPT case was presented in Ref. [29]. In the present problem, however, the main quantity of interest is the combination $(a_0^0 - a_0^2)$ of the scattering lengths, rather than the general scattering amplitude itself. Our aim is to establish a relationship between the pionium lifetime and this combination of the scattering lengths which could serve us to extract the value of $(a_0^0 - a_0^2)$, within an uncertainty interval, from the experimental value of the pionium lifetime. Furthermore, we want to place the prediction of standard χPT on the central line of the above relationship.

To achieve this, we have fixed the values of the parameters β and λ_i ($i = 1, \dots, 4$) from the predictions of standard χPT [3]. We have considered set I of threshold parameters, for which one has in particular $(a_0^0 - a_0^2) = 0.258$, and determined the parameters from the values of $a_0^0, a_0^2, a_1^1, a_2^0, b_0^2$ and b_1^1 . We have obtained the following values for the parameters: $\alpha = 1.021$, $\beta = 1.109$, $\lambda_1 = -10.28 \times 10^{-3}$, $\lambda_2 = 15.90 \times 10^{-3}$, $\lambda_3 = 0.81 \times 10^{-4}$, $\lambda_4 = -1.00 \times 10^{-4}$. These values should be considered as a convenient means of analyzing the present problem, rather than the best values for the entire scattering amplitude; their role is to introduce the standard χPT prediction as an initial data placed on the central line of the relationship between the lifetime and $(a_0^0 - a_0^2)$. Once the five parameters β and λ are fixed, a one-to-one correspondence is established between α and the combination $(a_0^0 - a_0^2)$ of the scattering lengths. Instead of considering α as the main variable of the problem, it is preferable to consider $(a_0^0 - a_0^2)$ as the principal variable of interest, since it has a direct physical meaning. We have thus considered a variation of $(a_0^0 - a_0^2)$ within the interval 0.250–0.370 which essentially covers the domain of generalized χPT , including standard χPT . For each given value of $(a_0^0 - a_0^2)$, the other combination $(2a_0^0 + a_0^2)$ is calculable through the low energy threshold parameter formulas [4].

The correction (28) to the decay width depends also on the electromagnetic low energy constants k^r . Some of these have been calculated in the $SU(3) \times SU(3)$ case in standard χPT in Ref. [28] and could be converted to the $SU(2) \times SU(2)$ case. We have assumed that the combination of the k^r s present in the standard χPT case (the terms in Eq. (24) not proportional to $(\alpha - \beta)$ or to $(\alpha - \beta)^2$) keeps its numerical value also in generalized χPT . As to the additional low energy constants (strong and electromagnetic) appearing

in generalized χPT , we have considered them as uncertainties of the order of $1/(16\pi^2)$ and added them quadratically. Furthermore, numerically $(\alpha)_0$ in Eq. (28) has been taken equal to α . Finally, the following numerical values have been used: $F_\pi = 93.2$ MeV, $m_{\pi^+} = 139.57$ MeV, $m_{\pi^0} = 134.97$ MeV. We have taken m_π , which appears in higher order corrective terms, equal to m_{π^+} and $\mu = m_\rho = 770$ MeV. [An error exists in the conversion formulas of Ref. [7] of the combination of k^r s of the standard χPT case, due to missing factors coming from the strong interaction low energy constants L^r and from the difference of chiral limits of F_π in the $SU(3) \times SU(3)$ and $SU(2) \times SU(2)$ cases. The correct formula has been given in Ref. [9]. Numerical predictions already made are however almost insensitive to this modification.]

The uncertainty in the relative correction (28) was estimated in Ref. [7], in the standard χPT case, to be of the order of 2% with respect to Γ_0 . This uncertainty has been maintained here and the additional one coming from the generalized χPT coefficients added to it. We have found that the uncertainty varies from 2% (standard χPT case) to 2.5% (extreme case of generalized χPT).

$a_0^0 - a_0^2$	α	a_0^0	$10a_0^2$	τ_0 (fs)	$(\Delta\tau/\tau_0)$	τ (fs)
0.250	0.77	0.205	-0.448	3.40	-0.066	3.20
0.254	0.89	0.211	-0.431	3.30	-0.063	3.11
0.258	1.02	0.217	-0.413	3.19	-0.061	3.03
0.262	1.13	0.222	-0.397	3.10	-0.058	2.94
0.270	1.37	0.234	-0.363	2.92	-0.054	2.78
0.290	1.95	0.262	-0.281	2.53	-0.043	2.44
0.300	2.23	0.276	-0.241	2.36	-0.038	2.29
0.310	2.51	0.290	-0.201	2.21	-0.034	2.16
0.320	2.78	0.304	-0.161	2.08	-0.029	2.03
0.330	3.05	0.318	-0.122	1.95	-0.025	1.92
0.340	3.31	0.332	-0.083	1.84	-0.021	1.81
0.350	3.57	0.346	-0.044	1.74	-0.018	1.72
0.360	3.83	0.360	-0.006	1.64	-0.014	1.63
0.370	4.09	0.373	+0.032	1.55	-0.011	1.55

Table 1: The ponium lifetime τ as a function of $a_0^0 - a_0^2$, with the low energy constants β and λ fixed from the set I solution of standard χPT [3]. τ_0 is the lifetime obtained from the nonrelativistic formula.

We present in Table 1 values of the ponium lifetime for some typical values of $(a_0^0 - a_0^2)$, as well as the corresponding values of α , a_0^0 , a_0^2 and the relative correction corresponding to Eq. (28).

We observe that the relative correction $|\Delta\tau/\tau_0|$ decreases with increasing α . This is mainly due to the fact that the tree level correction tends to zero as α approaches the value 4.

In Fig. 1 we have represented the curve of the lifetime τ as a function of the combination $(a_0^0 - a_0^2)$ of the scattering lengths (full line). The estimated uncertainties (2-2.5%) are represented by the band delineated by the dotted lines.

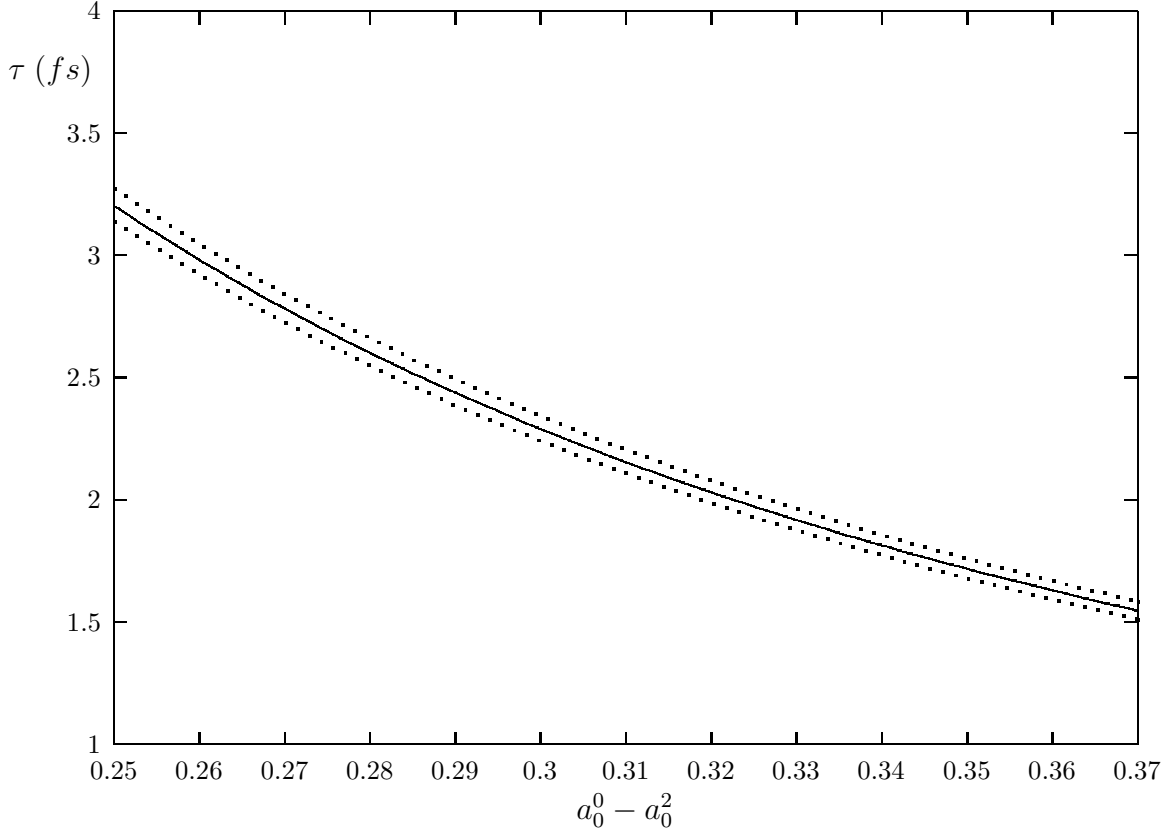


Figure 1: The pionium lifetime as a function of the combination $(a_0^0 - a_0^2)$ of the S -wave scattering lengths (full line). The band delineated by the dotted lines takes into account the estimated uncertainties (2-2.5%).

The theoretical value of the lifetime depends also on the uncertainties on the strong interaction threshold parameters, which were not considered above (these are not yet published in the literature in a definite form). A close analysis of the results obtained above shows, however, that eventual uncertainties on the threshold or low energy parameters have the tendency to move the predicted value of the lifetime by remaining within the band of uncertainty already considered. An illustration of this phenomenon can be seen by considering the set II solution obtained in the standard χPT case [3] and for which

one has in particular $(a_0^0 - a_0^2) = 0.250$. (A recent analysis of the threshold parameters in the standard χPT case is presented in Ref. [30].) One might for instance consider the differences between predictions of set I and set II as uncertainties. One then has two possibilities of proceeding. First, one can calculate directly from set II the value of the lifetime, by repeating the calculations done with set I. Second, one can seek from the curve of Fig. 1 the value of the lifetime corresponding to $(a_0^0 - a_0^2) = 0.250$. For the first method, we have found from the threshold parameters of set II the following values of the low energy constants: $\alpha = 1.010$, $\beta = 1.111$, $\lambda_1 = -9.11 \times 10^{-3}$, $\lambda_2 = 11.15 \times 10^{-3}$, $\lambda_3 = -1.14 \times 10^{-4}$, $\lambda_4 = -0.27 \times 10^{-4}$. The predicted value of the lifetime is $\tau = 3.22$ fs. This value lies within the band of uncertainty of Fig. 1. The value one obtains for $(a_0^0 - a_0^2) = 0.250$ from Fig. 1, corresponding to the initial data of set I, is $\tau = (3.20 \pm 0.07)$ fs, which underlines the fact that uncertainties of the order of 3% in the scattering length values have not moved the lifetime value outside the initial uncertainty band (and furthermore have left it rather close to the central line). Therefore, one should not enlarge further the initial uncertainty band of 2-2.5%, which could be considered as a conservative one.

One thus arrives at the conclusion that the relationship between the pionium lifetime and the combination $(a_0^0 - a_0^2)$ of the S -wave scattering lengths represented in Fig. 1, together with its uncertainty band, is practically independent of the particular choice of the low energy constants which yield the value of $(a_0^0 - a_0^2)$. In this sense, the curve of Fig. 1 should be used to determine from the experimental value of the pionium lifetime (with uncertainties) the corresponding value of $(a_0^0 - a_0^2)$ (with corresponding uncertainties). The determination of the other threshold parameters or low energy constants, and in particular of α and of the quark condensate parameter, requires a separate analysis with the aid of additional constraints, coming for instance from sum rules and the Roy equations.

It is also evident that for a resulting value of $(a_0^0 - a_0^2)$, because of the other existing constraints, the value of α , and hence of the quark condensate parameter, could not be arbitrarily varied. Independent of any detailed analysis which should determine the precise value of α , one still is allowed to derive from Fig. 1 some qualitative conclusions which we summarize as follows. Values of the lifetime close to 3 fs, lying above 2.9 fs, say, would confirm the scheme of standard χPT . Values of the lifetime lying below 2.4 fs remain outside the domain of predictions of standard χPT and would necessitate an alternative scheme of chiral symmetry breaking. Values of the lifetime lying in the interval 2.4-2.9 fs, because of the possibly existing uncertainties, would be more difficult to interpret and would require a more refined analysis; they might also indicate, within the standard χPT scheme, a strong dependence of the quark condensate on the light quark flavor number, as a trace of a possible chiral phase transition at higher values of the latter.

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